

Half-cycle soliton and periodic waves of arbitrarily high amplitude in a two-level system

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We show that the electromagnetic wave propagation problem based on a two-level atomic model admits exact traveling wave solutions, of both soliton and cnoidal wave types. These solutions extend the already known solutions of the same type of both the mKdV and sG equations, which have been derived from the same initial model, respectively, in a long-wave and in a short-wave approximation. The continuation is such that no modification of the wave profile is required, but that the wave velocity only has to be corrected.

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I. INTRODUCTION

Since the experimental achievement of optical pulses whose duration is close to a single optical cycle in 1999 [1–3], considerable effort has been made to propose an adequately renewed theoretical description of the wave propagation.

In the highest intensity range, the medium can be considered as fully ionized [4,5]. This is *a fortiori* true in the relativistic nonlinear optics regime [6], where the wave electric field may largely exceed the atomic field (the highest peak power yet achieved is close to 10 PW [7], which focused on a micrometer square would produce a intensities up to 10^{28} W/m²).

We will restrict ourselves here to the situation where the atomic structure is not destroyed by the light pulse. Within this limit, the main theoretical approaches that have been developed are the quantum one [8], the envelope approach, which uses generalizations of the nonlinear Schrödinger equation [9,10], and models proposed to completely avoid the use of the slowly varying envelope approximation (SVEA) [11].

Several non-SVEA models have been described, based on various descriptions of the medium. Some are macroscopic, such as the so-called short-pulse equation [12], and more complicated models [13,14]. Other models are based on a microscopic model of the atoms, which may be a classical one [15] or use the density matrix formalism [16,17].

However, all these studies use some low-amplitude, or “weakly nonlinear,” approximation, while the formation of few-cycle optical solitons requires in principle electric fields whose amplitude becomes comparable to that of the atoms.

Our goal in this paper is to propose a nonperturbative approach to optical wave propagation. We considered the simplest model of a material which can be written in the frame of the quantum mechanics: the two-level model in the density matrix formalism. Two asymptotic models have been derived from the two-level one beyond the SVEA, one being based on the modified Korteweg–de Vries (mKdV) equation, and the other on the sine-Gordon (sG) equation, depending if the central pulse frequency is well above or well below the frequency of the transition [18].

A more complicated model, assuming at least two transitions and containing both mKdV and sG-type contributions, has been shown to be very promising in the description of few-cycle-pulses (FCPs) propagation [19]. It has been shown that it is the most general model of this type [20]. Both the mKdV and the sG models can be generalized to many more realistic situations, taking into account the vector character of the field [21], the transverse dimensions of the space [22,23], and the more complex atomic structure of the material [24].

In the first section of this paper, we present the two-level model and derive the exact solution. In the second section, we compare this result to that of the mKdV model, which was derived in the long-wave approximation from the same starting point. In the third section, an analogous comparison is performed with the short-wave asymptotic, which is the sG model.

II. EXACT WAVE SOLUTIONS

A. The two-level model

Let us consider the Schrödinger–von Neumann equation

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho], \quad (1)$$

where ρ is the density matrix operator, and $H = H_0 - \mu \cdot \mathbf{E}$, with

$$H_0 = \hbar \begin{pmatrix} \omega_a & 0 \\ 0 & \omega_b \end{pmatrix}, \quad (2)$$

the two-level Hamiltonian, and

$$\mu = \begin{pmatrix} 0 & \mu \\ \mu^* & 0 \end{pmatrix} \mathbf{e}_x, \quad (3)$$

the atomic electric dipole moment operator, assumed to be directed along the direction defined by the unitary vector \mathbf{e}_x .

The evolution of the electric field \mathbf{E} is governed by the wave equation

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \frac{-1}{c^2} \frac{\partial^2}{\partial t^2} \left(\mathbf{E} + \frac{\mathbf{P}}{\epsilon_0} \right), \quad (4)$$

in which c is the speed of light in vacuum, ε_0 the dielectric permittivity in vacuum, and the polarization density \mathbf{P} is related to ρ by

$$\mathbf{P} = N\text{Tr}(\rho\mu). \quad (5)$$

We are looking for an exact progressive wave solution of these equations. We restrict ourselves to a linearly polarized wave and assume a planar wave propagation along z . Hence, we seek for a solution of the form

$$\mathbf{E} = E\left(t - \frac{z}{v}\right)\mathbf{e}_x \quad (6)$$

of Eqs. (1)–(5).

Obviously, we must have

$$\mathbf{P} = P\left(t - \frac{z}{v}\right)\mathbf{e}_x, \quad (7)$$

and (4) reduces to

$$\frac{-1}{v^2} \frac{d^2 E}{d\theta^2} = \frac{-1}{c^2} \frac{d^2}{d\theta^2} \left(E + \frac{P}{\varepsilon_0} \right), \quad (8)$$

where $\theta = t - z/v$ is the retarded time.

We are looking for either localized solutions E or periodic solutions with mean value zero. Hence P must have the same property, the integration constants must vanish, and Eq. (8) reduces to

$$P = \varepsilon_0 \chi E, \quad (9)$$

where we have defined the “susceptibility” χ by

$$\chi = \frac{c^2}{v^2} - 1. \quad (10)$$

We write the components of the density matrix as

$$\rho = \begin{pmatrix} \rho_a & \rho_t \\ \rho_t^* & \rho_b \end{pmatrix}, \quad (11)$$

and the expression of the polarization density (5) can be written as

$$P = N(\rho_t \mu^* + \rho_t^* \mu). \quad (12)$$

Replacing E by this expression of P divided by $\varepsilon_0 \chi$ in Eq. (1) we get the following set of equations:

$$i\hbar \frac{d\rho_a}{d\theta} = \frac{N}{\varepsilon_0 \chi} (\rho_t \mu^* + \rho_t^* \mu) (\rho_t \mu^* - \rho_t^* \mu), \quad (13)$$

$$i\hbar \frac{d\rho_t}{d\theta} = -\hbar(\omega_b - \omega_a)\rho_t - \frac{N}{\varepsilon_0 \chi} (\rho_t \mu^* + \rho_t^* \mu)(\rho_b - \rho_a)\mu, \quad (14)$$

$$i\hbar \frac{d\rho_b}{d\theta} = -\frac{N}{\varepsilon_0 \chi} (\rho_t \mu^* + \rho_t^* \mu)(\rho_t \mu^* - \rho_t^* \mu), \quad (15)$$

the fourth equation being the complex conjugate of (14).

We define

$$d = (\rho_t \mu^* + \rho_t^* \mu), \quad (16)$$

which is the atomic transition dipole moment, so that $P = Nd$, and

$$q = (\rho_t \mu^* - \rho_t^* \mu), \quad (17)$$

so that $2\rho_t \mu^* = d + iq$, with d and q real. $\Omega = (\omega_b - \omega_a)$ is the transition angular frequency,

$$w = (\rho_b - \rho_a), \quad (18)$$

is the population difference, negative in the considered situation, and $\rho_0 = (\rho_a + \rho_b)/2$.

Then it is seen that $d\rho_0/d\theta = 0$, in accordance with the property of the density matrix $\text{Tr}(\rho) = 1$, and the system reduces to

$$\frac{dw}{d\theta} = -2\gamma dq, \quad (19)$$

$$\frac{dd}{d\theta} = -\Omega q, \quad (20)$$

$$\frac{dq}{d\theta} = (\Omega + 2\gamma|\mu|^2 w)d, \quad (21)$$

where we have set

$$\gamma = \frac{N}{\varepsilon_0 \hbar \chi}, \quad (22)$$

for brevity. Equation (20) is solved straightforwardly as

$$q = \frac{-1}{\Omega} \frac{dd}{d\theta}. \quad (23)$$

Using (23) into Eq. (19) allows us to integrate it as

$$w = \frac{\gamma}{\Omega} d^2 - l, \quad (24)$$

where l is some constant. If E is localized, then d is as well, and $(-l)$ is the value of w in the absence of a wave. In this case, $\rho_b = \rho_{th}$ is the thermal excitation, and $\rho_a = 1 - \rho_{th}$, so that $l = 1 - 2\rho_{th}$, and $0 < l \leq 1$, where the equality occurs if all atoms are initially in the fundamental state. Since a periodic wave is a mathematical limit of a very long wave packet, the same interpretation still holds in this case.

Using the expression (23) of q into Eq. (21) yields an equation for d :

$$\frac{d^2 d}{d\theta^2} = Fd - 2Gd^3, \quad (25)$$

where we have set

$$F = (2l\gamma|\mu|^2 - \Omega)\Omega \quad (26)$$

and

$$G = \gamma^2 |\mu|^2. \quad (27)$$

Multiplying Eq. (25) by $dd/d\theta$ and integrating yield

$$\left(\frac{dd}{d\theta}\right)^2 = Fd^2 - Gd^4 + C, \quad (28)$$

where C is a constant, and finally

$$\theta = \pm \int \frac{dd}{\sqrt{C + Fd^2 - Gd^4}} + \theta_0, \quad (29)$$

θ_0 being another constant. It corresponds to translation invariance and the \pm sign to invariance by symmetry. We can restrict ourselves to the $+$ sign without loss of generality.

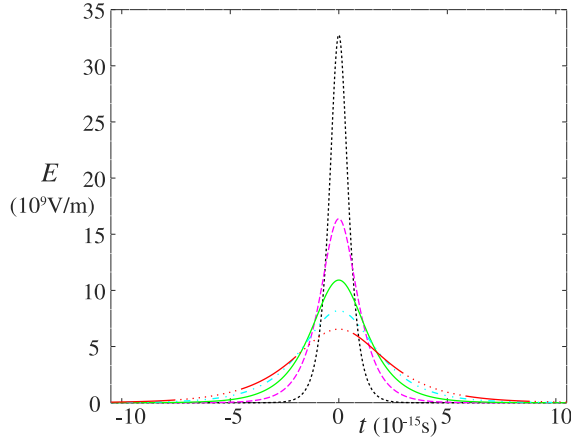


FIG. 1. Example of soliton profiles for several values of the soliton parameter, corresponding to pulse durations of $\Delta t = 1$ fs (dotted, black line), 2 fs (dashed, magenta), 3 fs (solid, green), 4 fs (one dash, two dots, cyan), and 5 fs (one dash, 10 dots, red).

B. The soliton

If the wave is localized, d and $dd/d\theta$ must vanish at infinity and $C = 0$. Then, and provided that F is positive, the integral can be computed explicitly, which yields

$$\theta - \theta_0 = \frac{-1}{\sqrt{F}} \ln \left[\frac{2}{|d|} (1 + \sqrt{1 - Gd^2/F}) \right], \quad (30)$$

which in turn can be inverted, as

$$d = \sqrt{\frac{F}{G}} \frac{1}{\cosh(\theta\sqrt{F})}, \quad (31)$$

after we have set $\theta_0 = \ln(2\sqrt{G/F})$ to simplify the expression.

However, the usual optical amplitude is that of the electric field, say, E_m . Since E is related to the polarization density $P = Nd$ through (9), we obtain

$$E = E_m \operatorname{sech} p \left(t - \frac{z}{v} \right), \quad (32)$$

where we have set $p = \sqrt{F}$, and the amplitude is

$$E_m = \frac{N}{\varepsilon_0 \chi} \sqrt{\frac{F}{G}}. \quad (33)$$

Using the definition (27) of G and the expression (22) of γ , E_m reduces to

$$E_m = \frac{\hbar}{|\mu|} p. \quad (34)$$

The velocity v of the soliton (32) is related to the soliton parameter $p = \sqrt{F}$ through the expression (26) of F and (22). We find

$$\chi = \frac{c^2}{v^2} - 1 = \frac{2IN|\mu|^2\Omega}{\varepsilon_0\hbar(\Omega^2 + p^2)}. \quad (35)$$

We plotted examples of the soliton profile in Fig. 1. The typical order of magnitude of an atomic dipole moment is $\mu = ea_0$, where $a_0 = 4\pi\varepsilon_0\hbar^2/(me^2)$ is the Bohr radius (with m and $-e$ the mass and electric charge of the electron); we use this value for the transition dipole moment.

We write the soliton parameter as $p = 2 \operatorname{acosh}(2)/\Delta t$, where Δt is the full width at half maximum of the pulse. Notice that the wave profile depends neither on the atomic density nor on the resonance frequency; these quantities affect only the velocity v .

C. The periodic solution

1. The cnoidal wave

We set for convenience

$$Q = C + Fd^2 - Gd^4. \quad (36)$$

When $C < 0$, the dipole moment d never can take the value zero in (29), which excludes that the corresponding solution can represent a wave oscillating around zero. However, for $C > 0$, the two roots of Q have opposite signs. Let us denote by d_m^2 the positive root. It is clear that

$$d_m^2 = \frac{F + \sqrt{F^2 + 4GC}}{2G} \quad (37)$$

and that d will oscillate between $-d_m$ and $+d_m$ with a period

$$T = 2 \int_{-d_m}^{d_m} \frac{dd}{\sqrt{Q}}. \quad (38)$$

In fact, the integral (29) can be computed explicitly using Jacobi elliptic functions [25]. Let us first write Q in normalized form by setting

$$\xi = \frac{4GC}{F^2}, \quad (39)$$

$\varepsilon = \operatorname{sgn}(F)$, and $Y^2 = 2Gd^2/|F|$, so that we can write Q as $Q = \Phi F^2/(4G)$, with

$$\Phi = x + 2\varepsilon Y^2 - Y^4. \quad (40)$$

The roots of Φ are

$$R_{\pm} = \varepsilon \pm \sqrt{1 + \xi} \quad (41)$$

and

$$\theta = \sqrt{\frac{2}{|F|}} \int \frac{dY}{\sqrt{\Phi}} + \theta_0. \quad (42)$$

Setting $Y = s\sqrt{R_+}$, and defining k by

$$\frac{1}{k^2} = 1 + \frac{\sqrt{1 + \xi} - \varepsilon}{\sqrt{1 + \xi} + \varepsilon}, \quad (43)$$

reduces Φ to

$$\Phi = \frac{R_+^2}{k^2} (1 - s^2)(1 - k^2 + k^2 s^2). \quad (44)$$

Since [25]

$$\int_{\sigma}^1 \frac{ds}{\sqrt{(1 - s^2)(1 - k^2 + k^2 s^2)}} = \operatorname{arccn}(\sigma, k), \quad (45)$$

we obtain, after some computation to get back to the initial variables and an adequate choice of the integration constant θ_0 ,

$$d = d_m \operatorname{cn} \left[p \left(t - \frac{z}{v} \right), k \right], \quad (46)$$

where k is given by (43),

$$d_m = \sqrt{\frac{|F|R_+}{2G}} \quad (47)$$

and

$$p = \frac{1}{k} \sqrt{\frac{|F|R_+}{2}}. \quad (48)$$

Using (41) and (39) allows us to show that the expression (47) of d_m exactly coincides with (37). Using in addition (43) allows us to reduce the expression (48) of p to

$$p = \sqrt{|F|(1+\xi)^{1/4}} = (F^2 + 4GC)^{1/4}. \quad (49)$$

Using (9), we see that the corresponding electric field is

$$E = E_m \text{cn} \left(p \left[t - \frac{z}{v} \right], k \right), \quad (50)$$

with

$$E_m = \frac{N}{\epsilon_0 \chi} d_m. \quad (51)$$

Since according to (47) and (48), $d_m = pk/\sqrt{G}$, it is seen using (27) and (22) that the amplitude E_m reduces to

$$E_m = \frac{\hbar}{|\mu|} kp. \quad (52)$$

Using the expression (49) of p , we see that the wave period is

$$T = \frac{4K(k)}{(F^2 + 4GC)^{1/4}}, \quad (53)$$

where $K(k)$ is the elliptic integral of the first kind. The angular frequency is thus

$$\omega = \frac{\pi p}{2K(k)}. \quad (54)$$

Using (54) in (52) we can write

$$E_m = \frac{2\hbar\omega}{\pi|\mu|} kK(k). \quad (55)$$

Finally, k is characterized by

$$kK(k) = \frac{\pi|\mu|E_m}{2\hbar\omega}, \quad (56)$$

and then

$$E = E_m \text{cn} \left(\frac{2\omega K(k)}{\pi} \left[t - \frac{z}{v} \right], k \right). \quad (57)$$

For the physical interpretation of the results, it would be more convenient to use the amplitude E_m of the wave and its angular frequency $\omega = 2\pi/T$ as parameters. However, this cannot be done explicitly except for asymptotic expressions.

2. The dispersion relation and the wave velocity

The dispersion relation is found by inverting the relations between the elliptic modulus k and the parameters of the system. Equation (43) can be simplified to yield

$$\frac{1}{k^2} = \frac{2}{\xi} \sqrt{1+\xi} (\sqrt{1+\xi} - \varepsilon), \quad (58)$$

which can be inverted to give the normalized parameter ξ [Eq. (39)] as

$$\xi = \frac{4k^2(1-k^2)}{(1-2k^2)^2}. \quad (59)$$

Then reporting (59) into (49) gives

$$p = \sqrt{\left| \frac{F}{1-2k^2} \right|}, \quad (60)$$

which can be inverted to yield

$$F = p^2(2k^2 - 1). \quad (61)$$

The correct sign is determined using the limit of (59) as ξ tends to zero: if $\varepsilon = +1$ ($F > 0$), then k tends to 1; if $\varepsilon = -1$ ($F < 0$), then k tends to 0. Owing to the definition (26) of F , Eq. (61) gives the velocity v of the cnoidal wave as a function of p and k , i.e., it is the dispersion relation. Making use of (22), we find

$$\chi = \frac{c^2}{v^2} - 1 = \frac{2IN|\mu|^2\Omega}{\epsilon_0\hbar[\Omega^2 + p^2(2k^2 - 1)]}. \quad (62)$$

It is known that for $k = 1$, $\text{cn}(X, k) = \text{sech}(X)$. It is easily checked that in this case the cnoidal wave (50) exactly coincides with the soliton (32), and that the velocity (62) of the cnoidal wave reduces to the velocity (35) of the soliton as $k = 1$.

For small values of the elliptic modulus, $k \ll 1$, since $\text{cn}(X, 0) = \cos(X)$, the cnoidal wave (46) becomes a sinusoidal wave, and we retrieve the linear limit. With a little higher values of k , we should also retrieve the so-called weakly nonlinear limit, that is, the nonlinear optics valid for picosecond pulses and intensities on the order of one GW/cm². This may result in a discussion about the high-intensity evolution of the susceptibility χ , which will be considered in further publication. We focus here on the agreement between the exact solution and asymptotic models derived from the same initial physical system in the study of FCP propagation.

We plotted examples of cnoidal wave profiles in Fig. 2, with the same value of the transition dipole moment as for Fig. 1. We write the wave frequency as $\omega = 2\pi c/\lambda$ and choose $\lambda = 0.5 \mu\text{m}$ for the figure. As above, the velocity v depends only on N and Ω . However, the relation between the wave intensity and the amplitude E_m involves v , hence the value of the elliptic modulus that corresponds to a given intensity also depends on the wave velocity.

III. COMPARISON WITH THE mKdV MODEL

The mKdV soliton

The mKdV model as derived in Ref. [18] can be written as

$$\frac{\partial u}{\partial \zeta} + 2 \frac{\partial u^3}{\partial \tau} + \frac{\partial^3 u}{\partial \tau^3} = 0, \quad (63)$$

with

$$\tau = \epsilon\Omega \left(t - \frac{z}{v_0} \right), \quad (64)$$

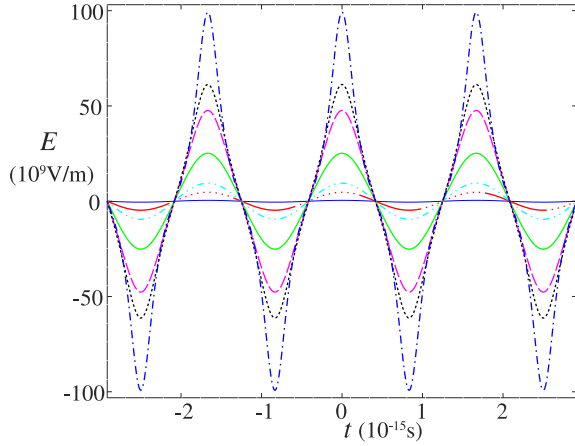


FIG. 2. Example of cnoidal wave profiles for $\lambda = 0.5 \mu\text{m}$ and several values of the elliptic modulus $k = 0.99$ (dash-dotted, blue line), 0.9 (dotted, black), 0.8 (dashed, magenta), 0.5 (solid, green), 0.2 (1 dash, 2 dots, cyan), and 0.1 (1 dash, 10 dots, red).

$$\zeta = \frac{-\epsilon^3 N |\mu|^2 v_0}{\epsilon_0 \hbar c^2} z, \quad (65)$$

$$u = \frac{|\mu|}{\epsilon \hbar \Omega} E, \quad (66)$$

in SI units, where v_0 , defined by

$$\frac{c^2}{v_0^2} = 1 + \frac{2N |\mu|^2}{\epsilon_0 \hbar \Omega^2}, \quad (67)$$

is the linear velocity at zero frequency, and ϵ the small perturbation parameter used to derive the mKdV model.

The mKdV equation (63) was shown to be completely integrable by means of the inverse scattering transform in Ref. [26], and its Hirota bilinear form was derived in Ref. [27]. Its fundamental soliton solution is

$$u = a \operatorname{sech}(a\tau - a^3 \zeta), \quad (68)$$

where a is the soliton parameter. Coming back to the laboratory coordinates, it is found, setting $p = \epsilon \Omega a$, that (68) coincides with the exact soliton (32) of the system, except that the velocity is v_1 defined by

$$\frac{1}{v_1} = \frac{1}{v_0} - \frac{N |\mu|^2 v_0 p^3}{\epsilon_0 \hbar c^2 \Omega^2}, \quad (69)$$

instead of the velocity v of the exact soliton, given by (35).

The mKdV equation (63) is derived under both a long-wave and a weak-amplitude assumption, which formally correspond to values of p small with respect to Ω (the assumption $\epsilon \ll 1$). In the expression (32) of the exact soliton, small values of p correspond indeed to both a low-amplitude and a long-wave approximation. The amplitude E_m given by (33) is proportional to p while the pulse duration is $1/p$, which becomes large in the considered limit.

Further, if this duration is associated to some photon with angular frequency $\omega_s = p$, (33) can be rewritten as $\hbar \omega_s = |\mu| E_m$, which expresses the equality between the photon energy $\hbar \omega_s$ and the interaction energy between the wave and an atom $E_m |\mu|$.

Since $p = \sqrt{F}$, small values of p correspond to small values of F . According its expression (26), this can occur either if the resonance frequency Ω can be considered as small or if $2l\gamma |\mu|^2$ is close to Ω . The former condition would, in contrast to the long-wave approximation used for mKdV, pertain to a situation where the actual wave frequency is large with respect to the resonance frequency Ω , i.e., a short-wave approximation. We are thus concerned here by the latter condition, which can be written using (22) as

$$\chi \simeq \frac{2lN |\mu|^2}{\epsilon_0 \hbar \Omega}, \quad (70)$$

which means that the soliton velocity v is close to the linear velocity at zero frequency v_0 [see (67)].

From the assumption $p \ll \Omega$, the factor $1/(\Omega^2 + p^2)$ in (35) can be written as $\Omega^{-2}(1 - 2p^2/\Omega^2)$, and it is found that

$$\frac{c}{v} = \left(\frac{c^2}{v_0^2} - \frac{2lN |\mu|^2 p^2}{\epsilon_0 \hbar \Omega^3} \right)^{1/2}. \quad (71)$$

Setting $l = 1$ since it was assumed in Ref. [18] that all atoms were initially in the fundamental state and performing a Taylor expansion of this expression show that the speed v of the exact soliton coincides with the speed v_1 of the fundamental soliton of the mKdV model in this limit.

Hence the fundamental soliton to the mKdV equation remains a solution to the complete model, and not only in the weak amplitude and long-wave approximation, provided that the velocity is corrected.

This can be related to the fully integrable mKdV hierarchy. It was established, in the case of the KdV equation which presents a mathematical equivalence with mKdV [28], applied to a propagation mode of electromagnetic polaritons in ferromagnetic media, that taking into account higher order terms in the asymptotic expansion that leads to KdV only modifies the velocity of the solitons, in all the domains where the asymptotic expansion series converges [29].

The same phenomenon occurs here, and we can expect that many other solutions provided by the mKdV model are in fact valid in a much wider range than the range where its derivation is valid.

Cnoidal wave of the mKdV equation

The cnoidal wave solution to the mKdV equation (63) is [30]

$$u = ka \operatorname{cn}(a\tau - a^3[2k^2 - 1]\zeta, k). \quad (72)$$

Using (64), (65), and (66), we come back to the laboratory coordinates, and, setting $p = \epsilon \Omega a$, it is found that the cnoidal wave solution (72) of mKdV coincides with the exact cnoidal wave (50)–(52) of the system, except that its velocity is v_1 defined by

$$\frac{1}{v_1} = \frac{1}{v_0} - \frac{lN |\mu|^2 v_0}{\epsilon_0 \hbar \Omega^3 c^2} p^2 (2k^2 - 1), \quad (73)$$

where v_0 is the linear velocity in the zero-frequency limit (67).

The velocity v of the exact cnoidal wave is given by (62). As in the case of the soliton, we expand it in a Taylor series of the quantity $p^2(2k^2 - 1)$, assumed to be very small with respect to Ω^2 , and we see that v exactly coincides with v_1

at the first order in this quantity. Hence, the exact cnoidal wave generalizes the mKdV approximation to arbitrary high amplitudes.

According to (52) and (54), it is seen that small values of p correspond to both small amplitudes E_m and small angular frequencies ω . According to (61), small values of p also correspond to small values of F , with any finite value of ξ . As in the case of solitons, small values of F imply that the wave velocity v is close to v_0 as is assumed by the mKdV model.

We see from Eq. (56) that the ratio $|\mu|E_m/\hbar\omega$ between the atom-wave interaction energy and the photon energy $\hbar\omega_s$ is $2kK(k)/\pi$. As k tends to zero, the quarter period $K(k)$ tends to $\pi/2$ while, as k tends to one, it diverges logarithmically as

$$K(k) \sim \ln \frac{4}{\sqrt{1-k^2}}, \quad (74)$$

hence $kK(k)$ can take any positive value, and consequently so does the ratio $|\mu|E_m/\hbar\omega$.

IV. COMPARISON WITH sG

A. The sG soliton

In Ref. [18] was developed a high-amplitude short-wave asymptotic to the Maxwell–von Neumann system, which yields a sG model for FCPs. A two-dimensional version of it was given in Ref. [23]. The sG model reads as

$$\frac{\partial^2 \psi}{\partial z \partial T} = U \sin \psi, \quad (75)$$

with

$$U = \frac{-lN\Omega|\mu|^2}{\varepsilon_0 \hbar c}, \quad (76)$$

$$\frac{\partial \psi}{\partial T} = \frac{2|\mu|}{\hbar} E, \quad (77)$$

and $T = t - z/c$. The fundamental soliton of the sG equation (75) can be written as [31]

$$\frac{\partial \psi}{\partial T} = 2p \operatorname{sech} p \left(T + \frac{U}{p^2} z \right), \quad (78)$$

which gives in physical units

$$E = \frac{\hbar p}{|\mu|} \operatorname{sech} p \left(t - \frac{z}{v_2} \right), \quad (79)$$

where the velocity v_2 is defined by

$$\frac{c}{v_2} = 1 + \frac{lN\Omega|\mu|^2}{\varepsilon_0 \hbar p^2}. \quad (80)$$

Expression (79) of the sG fundamental soliton exactly coincides with the expression (32) of the exact soliton, except that the velocity is v_2 , given by (80), instead of the velocity v given by (35).

The short-wave approximation used to derive the sG model assumes that the wave duration $1/p$ of the fundamental soliton is very small with respect to the optical period $2\pi/\Omega$ associated with the transition, hence that $p \gg \Omega$. Using this assumption, i.e., neglecting Ω^2 with respect to p^2 , in (35), we see that v coincides with v_2 in this limit.

B. Cnoidal wave

The sG equation (75) also admits the exact periodic solution [32]

$$\psi = 2 \operatorname{asin} \left\{ k \operatorname{sn} \left[p \left(T - \frac{U}{p^2} z \right), k \right] \right\}. \quad (81)$$

According to (77), this yields an electric field

$$E = \frac{\hbar k p}{|\mu|} \operatorname{cn} \left[p \left(t - \frac{z}{v'_2} \right), k \right] \quad (82)$$

with some velocity v'_2 , defined by

$$\frac{1}{v'_2} = \frac{1}{c} + \frac{U}{p^2}. \quad (83)$$

v'_2 should be compared to the velocity v of the exact cnoidal wave as given by (62). However, expression (76) of the constant U cannot be used here, because it is derived under the assumption that the wave is localized, which is not valid anymore.

In Ref. [23] it was found that the two quantities $A = w/w_r$ with

$$w_r = \frac{2\varepsilon_0 \hbar c}{N\Omega|\mu|^2} \quad (84)$$

and

$$B = \frac{2|\mu|}{\hbar} E \quad (85)$$

satisfy the two equations

$$\frac{\partial^2 B}{\partial z \partial T} = AB, \quad (86)$$

$$\frac{\partial A}{\partial T} = -B \frac{\partial B}{\partial z}. \quad (87)$$

Then, following an equivalence between system (86)–(87) and the sG equation first evidenced in Ref. [33], the change of variables

$$A = U \cos \psi, \quad \frac{\partial B}{\partial z} = U \sin \psi \quad (88)$$

was introduced, and it was seen that U must be constant with respect to T and that ψ satisfies the sG equation (75). Further, $\psi = \partial B / \partial z$, according to (77).

With the assumption that the wave is localized, both E and its derivative vanish at infinity, while the population difference w tends to its thermal value $-l$, which gives the expression (76) of U . This does not remain true with the periodic solution (81). The explicit computation of A can be performed by integrating (87), as

$$A = \frac{w'}{2} B^2 + A_0, \quad (89)$$

where A_0 is some constant. Substituting (81) in (89) and (85) allows us to compute explicitly $U^2 = A^2 + (\partial B / \partial z)^2$. It is found after some computation that U^2 expresses as a polynomial function of $[\operatorname{cn} p(T - w'z)]^2$, and that this polynomial is a constant only if

$$A_0 = -U(2k^2 - 1), \quad (90)$$

which is related to both the parameter U of the sG equation (75) and the elliptic modulus k . Then reporting the expression (24) of w into A , it is found that $A_0 = -l/w_r$, which gives

$$U = \frac{lN|\mu|^2\Omega}{\varepsilon_0\hbar c(2k^2 - 1)}. \quad (91)$$

Reporting (91) into (83), and taking into account the fact that both the relative inverse velocity U/p^2 of the cnoidal wave and Ω are assumed to be small, it is seen that v'_2 coincides with the velocity v of the exact cnoidal wave given by (62).

Hence the exact cnoidal wave generalizes that of the sG model. More precisely, as in the case of the mKdV model, the expressions of both the soliton and the cnoidal wave solutions to the sG model remain valid for an arbitrary amplitude, provided that the velocity is adequately modified.

V. CONCLUSION

We can summarize the above results as follows: the electromagnetic wave propagation problem based on a two-level model of the atoms admits exact traveling wave solutions, of both soliton and cnoidal wave types. These solutions extend the already known solutions of the same type of both the mKdV and the sG models, which have been derived in a long-wave and in a short-wave approximation of the same basic equations, respectively. The continuation is such that no modification of the wave profile is required, but that the wave speed only has to be corrected. Hence, insofar that the two-level model is valid, either the mKdV or the sG model, with adequate modification of the dispersion, would accurately describe the traveling waves.

Here we mean by traveling waves those that remain exactly unchanged during propagation. We know that FCPs are rather of the breather soliton type [19], that is, that they oscillate periodically during the propagation. To clarify this using linear (or weakly nonlinear) concepts, the pulse envelope remains unchanged, but the exact wave profile oscillates due to the mismatch between the group and the phase velocities. The breather solitons of both the mKdV and sG equations are built

from an analytic continuation of their two-soliton solutions [19]; hence, in some sense, they are obtained by nonlinear superposition of two solitons of the type considered in the present paper. Consequently, we can expect that a mKdV or sG model with adequate modification of the dispersion would also be able to describe quite accurately such solutions far beyond the validity domain of the long- or short-wave approximation under which it has been derived.

Further, it has been shown that the mKdV model remains valid when the Hamiltonian contains an arbitrary number of atomic levels [24]. Hence we can expect that the above conclusion, that the validity of the mKdV model can be widely extended in the case of a two-level model, would remain valid for a multilevel model.

The validity of the sG model does not extend as easily, because the sine term takes into account the population difference corresponding to the atomic transition and, consequently, as many sine terms are required as transitions are considered [34]. However, the behavior of the double sG equation does not so essentially differ from that of the sG equation itself, and we can expect that a small number of terms would allow reasonable accuracy to be obtained in any case.

Finally, we can conclude that, provided that no resonant atomic process is involved, and that adequate corrections to the dispersion terms are added, the validity of the mKdV-sG-type models is limited only by that of the quantum model with a finite number of atomic levels, that is, by the fact that it does not take into account the possible ionization of the wave.

These models, however, concern optical pulses with a huge intensity, so that the electric field of the wave is comparable to the atomic field. A large part of the nonlinear optics is performed at more reasonable intensities, using models of the nonlinear Schrödinger (NLS) type. The question, if the exact solutions derived here can be considered as generalizations of solutions to the NLS model, is of obvious interest. Also relevant would be their interpretation in terms of nonlinear phase shift and harmonics generation. However, the agreement between the exact solution and the usual theory is not obvious and requires a long discussion, which is left for future publication.

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- [1] U. Morgner, F. X. Kärtner, S. H. Cho, Y. Chen, H. A. Haus, J. G. Fujimoto, E. P. Ippen, V. Scheuer, G. Angelow, T. Tschudi, Sub-two-cycle pulses from a Kerr-lens mode-locked Ti:sapphire laser, *Opt. Lett.* **24**, 411 (1999).
 - [2] D. H. Sutter, G. Steinmeyer, L. Gallmann, N. Matuschek, F. Morier-Genoud, U. Keller, V. Scheuer, G. Angelow, and T. Tschudi, Semiconductor saturable-absorber mirror-assisted Kerr-lens mode-locked Ti:sapphire laser producing pulses in the two-cycle regime, *Opt. Lett.* **24**, 631 (1999).
 - [3] T. Brabek and F. Krausz, Intense few-cycle laser fields: Frontiers of nonlinear optics, *Rev. Modern Phys.* **72**, 545 (2000).
 - [4] L. Bergé, S. Skupin, R. Nuter, J. Kasparian, and J.-P. Wolf, Ultrashort filaments of light in weakly ionized, optically transparent media, *Rep. Prog. Phys.* **70**, 1633 (2007).
 - [5] C. J. Joshi and P. B. Corkum, Interactions of ultra-intense laser light with matter, *Phys. Today* **48**(1), 36 (1995).
 - [6] G. A. Mourou, T. Tajima, and S. V. Bulanov, Optics in the relativistic regime, *Rev. Modern Phys.* **78**, 309 (2006).
 - [7] W. Li, Z. Gan, L. Yu, C. Wang, Y. Liu, Z. Guo, L. Xu, M. Xu, Y. Hang, Y. Xu, J. Wang, P. Huang, H. Cao, B. Yao, X. Zhang, L. Chen, Y. Tang, S. Li, X. Liu, S. Li, M. He, D. Yin, X. Liang, Y. Leng, R. Li, and Z. Xu, 339 J high-energy Ti:sapphire chirped-pulse amplifier for 10 PW laser facility, *Opt. Lett.* **43**, 5681 (2018).
 - [8] A. Nazarkin, Nonlinear Optics of Intense Attosecond Light Pulses, *Phys. Rev. Lett.* **97**, 163904 (2006).
 - [9] T. Brabek and F. Krausz, Nonlinear Optical Pulse Propagation in the Single-Cycle Regime, *Phys. Rev. Lett.* **78**, 3282 (1997).
 - [10] M. V. Tognetti and H. M. Crespo, Sub-two-cycle soliton-effect pulse compression at 800 nm in photonic crystal fibers, *J. Opt. Soc. Am. B* **24**, 1410 (2007).

- [11] H. Leblond and D. Mihalache, Models of few optical cycle solitons beyond the slowly varying envelope approximation, *Phys. Rep.* **523**, 61 (2013).
- [12] T. Schäfer and C. E. Wayne, Propagation of ultra-short optical pulses in cubic nonlinear media, *Physica D* **196**, 90 (2004).
- [13] V. G. Bespalov, S. A. Kozlov, Yu. A. Shpolyanskiy, and I. A. Walmsley, Simplified field wave equations for the nonlinear propagation of extremely short light pulses, *Phys. Rev. A* **66**, 013811 (2002).
- [14] Sh. Amiranashvili, A. G. Vladimirov, and U. Bandelow, Solitary-wave solutions for few-cycle optical pulses, *Phys. Rev. A* **77**, 063821 (2008).
- [15] E. V. Kazantseva, A. I. Maimistov, and J.-G. Caputo, Reduced Maxwell-Duffing description of extremely short pulses in non-resonant media, *Phys. Rev. E* **71**, 056622 (2005).
- [16] I. V. Mel'nikov, D. Mihalache, F. Moldoveanu, and N.-C. Panoiu, Quasiadiabatic following of femtosecond optical pulses in a weakly excited semiconductor, *Phys. Rev. A* **56**, 1569 (1997).
- [17] S. V. Sazonov, Extremely short and quasi-monochromatic electromagnetic solitons in a two-component medium, *JETP* **92**, 361 (2001).
- [18] H. Leblond and F. Sanchez, Models for optical solitons in the two-cycle regime, *Phys. Rev. A* **67**, 013804 (2003).
- [19] H. Leblond, S. V. Sazonov, I. V. Mel'nikov, D. Mihalache, and F. Sanchez, Few-cycle nonlinear optics of multicomponent media, *Phys. Rev. A* **74**, 063815 (2006).
- [20] H. Leblond and D. Mihalache, Few-optical-cycle solitons: Modified Korteweg-de Vries sine-Gordon equation versus other non-slowly-varying-envelope-approximation models, *Phys. Rev. A* **79**, 063835 (2009).
- [21] H. Leblond, H. Triki, F. Sanchez, and D. Mihalache, Robust circularly polarized few-optical-cycle solitons in Kerr media, *Phys. Rev. A* **83**, 063802 (2011).
- [22] H. Leblond, D. Kremer, and D. Mihalache, Collapse of ultra-short spatiotemporal pulses described by the cubic generalized Kadomtsev-Petviashvili equation, *Phys. Rev. A* **81**, 033824 (2010).
- [23] H. Leblond and D. Mihalache, Ultrashort light bullets described by the two-dimensional sine-Gordon equation, *Phys. Rev. A* **81**, 063815 (2010).
- [24] H. Triki, H. Leblond, and D. Mihalache, Derivation of a modified Korteweg-de Vries model for few-optical-cycles soliton propagation from a general Hamiltonian, *Opt. Commun.* **285**, 3179 (2012).
- [25] J. V. Armitage and W. F. Eberlein, *Elliptic Functions* (Cambridge University Press, Cambridge, 2006).
- [26] M. Wadati, The modified Korteweg-de Vries equation, *J. Phys. Soc. Jpn.* **34**, 1289 (1973).
- [27] R. Hirota, Direct method of finding exact solutions of nonlinear evolution equations, in *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications (Workshop Contact Transformation, Vanderbilt Univ., Nashville, Tenn., 1974)*, Lecture Notes in Mathematics Vol. 515 (Springer, Berlin, 1976), pp. 40–68.
- [28] R. M. Miura, Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, *J. Math. Phys.* **9**, 1202 (1968).
- [29] H. Leblond, A new criterion for the existence of KdV solitons in ferromagnets, *J. Phys. A* **36**, 1855 (2003).
- [30] C. F. Driscoll and T. M. O'Neil, Those ubiquitous, but oft unstable, lattice solitons, *Rocky Mount. J. Math.* **8**, 211 (1978).
- [31] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
- [32] F. Natali, On periodic waves for sine- and sinh-Gordon equations, *J. Math. Anal. Appl.* **379**, 334 (2011).
- [33] R. A. Kraenkel, M. A. Manna, and V. Merle, Nonlinear short-wave propagation in ferrites, *Phys. Rev. E* **61**, 976 (2000).
- [34] H. Leblond, H. Triki, and D. Mihalache, Derivation of a generalized double sine-Gordon equation describing ultrashort soliton propagation in optical media composed of multilevel atoms, *Phys. Rev. A* **86**, 063825 (2012).